Eur. Phys. J. B **13**, 531–537 (2000)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Non-equilibrium dynamics of simple spherical spin models

W. Zippold, R. Kühn[a], and H. Horner

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany

Received 22 April 1999

Abstract. We investigate the non-equilibrium dynamics of spherical spin models with two-spin interactions. For the exactly solvable models of the d-dimensional spherical ferromagnet and the spherical Sherrington-Kirkpatrick (SK) model the asymptotic dynamics has for large times and large waiting times the same formal structure. In the limit of large waiting times we find in both models an intermediate time scale, scaling as a power of the waiting time with an exponent smaller than one, and thus separating the time-translation-invariant short-time dynamics from the aging regime. It is this time scale on which the fluctuation-dissipation theorem is violated. Aging in these models is similar to that observed in spin glasses at the level of correlation functions, but different at the level of response functions, and thus different at the level of experimentally accessible quantities like thermoremanent magnetization.

PACS. 05.20.-y Classical statistical mechanics – 75.10.Nr Spin-glass and other random models

1 Introduction

There exist many systems which exhibit relaxation times long enough to keep them from reaching equilibrium on experimental time scales. Primary examples are spin glasses and polymer glasses, but also systems as simple as the Ising model when prepared in an arbitrary initial state, or phase separation dynamics in systems with conserved parameter such as Ostwald ripening in binary alloys. As a consequence, the relaxation depends for all these systems on the waiting time $t_{\rm w}$ already spent in the low temperature phase: the systems age. To understand the aging phenomena observed in these models one has to investigate their non-equilibrium dynamics.

In this context the investigation of the non-equilibrium dynamics of spherical spin models with two-spin interactions is interesting because they exhibit nontrivial dynamical behaviour, despite their simplicity, which makes their non-equilibrium dynamics exactly solvable. These systems never reach equilibrium, hence correlation and response functions depend on the waiting time even in the limit of large times [6,7,9].

Our main aim is to complete for this class of models the analysis of the spherical SK model presented in [9] by identifying *all* relevant time scales of the problem. We are going to show that, in addition to the two time regimes found in [9], there exists an intermediate time scale $t_{\rm p} \gg 1$ satisfying $t_{\rm p}/t_{\rm w} \to 0$ for $t_{\rm w} \to \infty$. It is this intermediate time scale on which the fluctuation dissipation theorem is beginning to be violated. Interest in this time scale stems from the fact that a thorough understanding of the dynamics at these intermediate times is important with respect to the study of the non-equilibrium dynamics in models more complicated than those considered in the present paper, such as that of the spherical p-spin glass with $p > 2$, because it is the behaviour at the time scale $t_{\rm p}$ which determines the behaviour at the time scale $t_{\rm w}$ in a unique way. It is thus the key ingredient towards the solution of the so far unsolved problem of selecting a unique solution within an infinite family of time reparametrization covariant solutions on diverging time scales, as has been demonstrated within a multi-domain crossover scaling approach for the closely related problem of a slowly dragged particle in a random potential [12]. The analysis of the simple spherical spin models is presented here, because their behaviour at the intermediate time scale can be studied analytically and in instructive detail.

Moreover we shall see that, despite the similarity of these models to the more difficult case of the spherical p-spin glass, their dynamics is not spin glass dynamics. This has been realized for some time from considerations concerning fluctuation dissipation ratios or parametric plots of an integrated response *versus* correlation (see *e.g.* [6,11]). Alternatively, one may look at the thermoremanent magnetization (another form of integrated response) as a quantity sensitive to the complicated phase space structure, to distinguish spin glasses from the simpler magnetic systems. While the thermoremanent magnetization, when plotted against logarithmic time, exhibits a waiting time dependent plateau in spin glasses, this plateau is absent in the models considered here.

We have organized our material as follows. In Section 2 we introduce the models and briefly review the general method for solving their non-equilibrium

[a] e-mail: kuehn@tphys.uni-heidelberg.de

dynamics, as first presented in [9]. In Section 3 we specialize to the spherical SK model and to d dimensional hyper-cubic spherical ferromagnets, which independently of the dimension d of the latter, exhibit formally the same type of long time non-equilibrium dynamics; exponents describing the decay of correlation and response for the latter vary, of course, with d. Time scales are identified and analyzed in Section 4, while Section 5 contains a discussion of our results.

2 The model

We consider spherical spin models with two spin interactions consisting of N continuous spins $s_i(t)$, $i = 1, \ldots, N$, which satisfy for all times t the spherical constraint $\sum_{i=1}^{N} {s_i}^2(t) = N$. The Hamiltonian of the system is given by

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j. \tag{1}$$

The coupling matrix J_{ij} is supposed to be an arbitrary symmetric matrix. Denoting the eigenvalues of the matrix J by a_i, $i = 1, \ldots, N$, the system of Langevin equations, which describes the dynamics of the model, decouples in terms of the projections $s_{a_i}(t)$ of the spins $s_i(t)$ onto the eigenvectors

$$\begin{aligned} \partial_t s_{a_i}(t) &= (a_i - \mu(t)) s_{a_i}(t) \\ &\quad + h_{a_i}(t) + \xi_{a_i}(t),\ i = 1, \ldots, N, \end{aligned} \tag{2}$$

where $h_{a_i}(t)$ is the corresponding component of an external magnetic field and $\xi_{a_i}(t)$ is thermal Gaussian white noise with zero mean and correlation $\langle \xi_{a_i}(\tau + t_{\mathrm{w}}) \xi_{a_j}(t_{\mathrm{w}}) \rangle = 2T \delta_{ij} \delta(\tau)$. The parameter $\mu(t)$ is the Lagrange multiplier enforcing the spherical constraint. Henceforth we will use $\langle \cdot \rangle$ to represent the average over the thermal noise. If it were not for the Lagrange parameter $\mu(t)$, the dynamics (2) would just be that of N independent harmonic oscillators under the influence of thermal noise. This means that solving the non-equilibrium dynamics of these models reduces to determining $\mu(t)$. It was shown in reference [9] that for a given waiting time t_{w} and given time separation $\tau \geq 0$ the autocorrelation $q(\tau, t_{\mathrm{w}}) := 1/N \left[\sum_{i=1}^{N} \langle s_i(\tau + t_{\mathrm{w}}) s_i(t_{\mathrm{w}}) \rangle \right]_J$ and response function $r(\tau, t_{\mathrm{w}}) := 1/N \left[\sum_{i=1}^{N} \delta \langle s_i(\tau + t_{\mathrm{w}}) \rangle / \delta h_i(t_{\mathrm{w}}) |_{h=0} \right]_J$ of this class of models are in terms of

$$\Lambda(t) := \exp \left(2 \int_0^t \mathrm{d}s\, \mu(s) \right) \tag{3}$$

given by

$$\begin{aligned} q(\tau, t_{\mathrm{w}}) = &\frac{\Lambda(t_{\mathrm{w}} + \tau/2)}{\sqrt{\Lambda(\tau + t_{\mathrm{w}}) \Lambda(t_{\mathrm{w}})}} \\ &\times \left[1 - T \int_0^\tau \mathrm{d}s\, \frac{\Lambda(t_{\mathrm{w}} + \tau/2 - s/2)}{\Lambda(t_{\mathrm{w}} + \tau/2)} \langle\langle \exp(as) \rangle\rangle \right] \end{aligned} \tag{4}$$

and

$$r(\tau, t_{\mathrm{w}}) = \sqrt{\frac{\Lambda(t_{\mathrm{w}})}{\Lambda(\tau + t_{\mathrm{w}})}} \langle\langle \exp(a\tau) \rangle\rangle, \tag{5}$$

where we have specialized the expressions in [9] to the case of zero external field and constant temperature T and have chosen the initial condition to be $s_{a_i}(t = 0) = 1$. By $[\cdot]_J$ we have denoted a possible disorder average and by $\langle\langle \cdot \rangle\rangle$ we denote the integration $\int \mathrm{d}a \rho(a) \cdot$ over an eigenvalue density $\rho(a)$ which in the thermodynamic limit $N \to \infty$ describes the distribution of eigenvalues of the coupling matrix J. The quantity $\Lambda(t)$ itself is determined by

$$\Lambda(t) = \langle\langle \exp(2at) \rangle\rangle + 2T \int_0^t \mathrm{d}s \Lambda(s) \langle\langle \exp(2a(t-s)) \rangle\rangle, \tag{6}$$

which together with (4) immediately implies $\Lambda(0) = q(0, t_{\mathrm{w}}) = 1$. Another dynamical observable we will be interested in is the thermoremanent magnetization $m_r(\tau, t_{\mathrm{w}})$. Given that the system is kept in a small magnetic field h in the time interval $[0, t_{\mathrm{w}}]$, the magnetization measured at time $\tau + t_{\mathrm{w}}$ is given by

$$m_r(\tau, t_{\mathrm{w}}) = h \int_0^{t_{\mathrm{w}}} \mathrm{d}s\, r(\tau + s, t_{\mathrm{w}} - s). \tag{7}$$

For models such as the d dimensional ferromagnets considered in what follows, in which the interaction matrix has a geometrical structure, off-diagonal correlations of the form $q_{ij}(\tau, t_{\mathrm{w}}) = \langle s_i(\tau + t_{\mathrm{w}}) s_j(t_{\mathrm{w}}) \rangle$ are of course also of interest. We have not investigated them in the present paper, however, as our main interest here is in results which will have further bearing on spin glass models of the mean field type.

3 Spherical SK model and spherical ferromagnet

While the expressions given so far are valid for any choice of the coupling matrix J we now want to treat two special cases. Our aim is to solve the non-equilibrium dynamics of these particular models in the limit of large waiting times $t_{\mathrm{w}} \gg 1$ by explicitly determining $\Lambda(t)$. As we are only interested in the behaviour of the dynamical observables for times $\tau, t_{\mathrm{w}} \gg 1$ it is sufficient to determine the asymptotic behaviour of $\Lambda(t)$ for $t \gg 1$. In the following we will discuss the d-dimensional spherical ferromagnet and the spherical SK model. The latter is the special case $p = 2$ of the disordered spherical p-spin model and we will present the results found in [9] in a slightly different form.

In the case of the spherical ferromagnet we consider a d-dimensional hyper-cubic lattice with periodic boundary conditions, whose lattice constant we take to be unity and whose lattice sites with coordinate vectors $\mathbf{x}_i$ are occupied by spins s_i. The couplings are chosen to be ferromagnetic nearest neighbour interactions, whose strength is set to unity. The standard diagonalization procedure using Fourier modes [3] yields for this choice of the matrix J in the limit $N \to \infty$ for the spectrum of eigenvalues and the eigenvalue density $\rho^{\mathrm{fm}}(a)$ the result

$$\rho^{\mathrm{fm}}(a) = \frac{1}{\pi}\int_0^\infty \mathrm{d}y \cos(ay)\,[J_0(2y)]^d, \quad a \in [-2d, 2d], \tag{8}$$

where $J_0(y)$ denotes the Bessel function of zeroth order.

The spherical SK model is defined by choosing the coupling matrix J to be a random matrix whose entries J_{ij} are independent and identically distributed Gaussian random variables with zero mean and variance $[(J_{ij})^2]_J = 1/N$. A general result of random matrix theory [17] states that, for this choice of J, the eigenvalue density in the thermodynamic limit ρ^{sk} is given by the Wigner semi-circle law

$$\rho^{\mathrm{sk}}(a) = \frac{1}{2\pi}\sqrt{4-a^2} \quad a \in [-2, 2]. \tag{9}$$

Solving the non-equilibrium dynamics of these models means solving (6) for the eigenvalue densities (8, 9). This is best done using the Laplace transform $\tilde{\Lambda}(s) = \int_0^\infty \mathrm{d}t \Lambda(t)\exp(-st)$ to obtain from (6) for $t > 0$ the relation

$$\tilde{\Lambda}(s) = \frac{\tilde{f}(s)}{1 - 2T\tilde{f}(s)}, \tag{10}$$

where the function $\tilde{f}(s) := \langle\langle 1/(s-2a)\rangle\rangle$ is characteristic of the given model. In terms of the function $f(t)$, which yields $\tilde{f}(s)$ *via* Laplace transformation, the expressions (4) and (5) can be rewritten as

$$q(\tau, t_{\mathrm{w}}) = \frac{\Lambda(t_{\mathrm{w}} + \tau/2)}{\sqrt{\Lambda(\tau + t_{\mathrm{w}})\Lambda(t_{\mathrm{w}})}} \times \left[1 - \frac{1}{2}T\int_0^{2\tau} \mathrm{d}x \frac{\Lambda(t_{\mathrm{w}} + \tau/2 - x/4)}{\Lambda(t_{\mathrm{w}} + \tau/2)}\, f(x/4)\right], \tag{11}$$

and

$$r(\tau, t_{\mathrm{w}}) = \sqrt{\frac{\Lambda(t_{\mathrm{w}})}{\Lambda(\tau + t_{\mathrm{w}})}}\, f(\tau/2), \tag{12}$$

respectively. Inserting the expressions for the eigenvalue densities (8) and (9) in the definition of $\tilde{f}(s)$ we find that in the case of the spherical ferromagnet the function $f(t)$ is given by

$$f^{\mathrm{fm}}(t) = [I_0(4t)]^d, \tag{13}$$

while for the spherical SK model it is calculated to be

$$f^{\mathrm{sk}}(t) = \frac{I_1(4t)}{2t}\,\cdot \tag{14}$$

In these expressions $I_0(t)$ and $I_1(t)$ denote the modified Bessel function of zeroth and first order, respectively.

The critical temperature of the dynamic phase transition is found from (10) to be

$$T_{\mathrm{c}} = \frac{1}{2\tilde{f}(2a_m)}, \tag{15}$$

where a_m denotes the maximal eigenvalue of the eigenvalue spectrum $[-a_m, a_m]$ of the corresponding model. In the special cases considered here we have $a_m^{\mathrm{fm}} = 2d$ and $a_m^{\mathrm{sk}} = 2$. Expression (15) implies in the case of the spherical ferromagnet that we get a phase transition in $d > 2$ only. For the spherical SK model the critical temperature can be calculated explicitly and one finds $T_{\mathrm{c}}^{\mathrm{sk}} = 1$. These results are in both models in agreement with the ones obtained from static calculations of the transition temperature [4,15].

In the following we will only be interested in the low temperature phase with $T < T_{\mathrm{c}}$. In this phase the asymptotic behaviour of $\Lambda(t)$ for large times t is determined by the behaviour of the Laplace transform $\tilde{\Lambda}(s)$ at the right bound $s = 2a_m$ of the branch cut. In the case of the spherical SK-model the inverse Laplace transformation can be done exactly and the result found in [9] reads

$$\Lambda^{\mathrm{sk}}(t) = \frac{1}{T}\sum_{k=1}^{\infty} kT^k \frac{I_k(4t)}{2t}\,\cdot \tag{16}$$

For the spherical ferromagnet there is no simple analytic expression, but we can find the leading asymptotic behaviour for $t \gg 1$ in integer dimension $d > 2$ by expanding $\tilde{\Lambda}(s)$ around $s = 2a_m$ and performing the inverse Laplace transformation. Performing this calculation and comparing the result with the leading order of expression (16) for large times $t \gg 1$, we find that in both models the leading asymptotic behaviour of $\Lambda(t)$ for $t \gg 1$ can be written as

$$\Lambda(t) \simeq \frac{\Lambda_0}{(1 - T/T_{\mathrm{c}})^2}\frac{\mathrm{e}^{2a_m t}}{t^{\nu_{\mathrm{s}}}} \quad t \gg 1, \tag{17}$$

where the prefactor Λ_0 is given by $\Lambda_0^{\mathrm{fm}} = (8\pi)^{-\nu_{\mathrm{s}}}$ for the spherical ferromagnet and by $\Lambda_0^{\mathrm{sk}} = (32\pi)^{1-\nu_{\mathrm{s}}}$ for the spherical SK model. The exponent ν_{s} appearing in these expressions is $\nu_{\mathrm{s}}^{\mathrm{fm}} = d/2$ for the spherical ferromagnet and $\nu_{\mathrm{s}}^{\mathrm{sk}} = 3/2$ for the spherical SK model. In the same way it follows from (13) and (14) that the asymptotic behaviour of $f(t)$ can in both models be written as

$$f(t) \simeq \Lambda_0 \frac{\mathrm{e}^{2a_m t}}{t^{\nu_{\mathrm{s}}}} \quad t \gg 1, \tag{18}$$

with the same factor Λ_0 as in (17). These two formulas indicate already the close correspondence of the asymptotic behaviour of autocorrelation and response function in both models which we will study in more detail in the following section. We will see that we can write the expressions for these dynamical observables for both models in a unified way using the exponent ν_{s} defined above. This means in particular that we will find the same time scales appearing in the limit of large waiting times t_{w} and large time separations τ.

4 Asymptotic dynamics and time scales

Before entering the discussion of the asymptotic behaviour for large waiting times t_w, we want to present for later reference the expressions for correlation and response function for finite waiting time $t_\mathrm{w} \sim 1$ and large times $\tau \gg 1$. Using (17) and (18) in the terms containing the time variable t in (11) and (12) we obtain for the autocorrelation

$$q(\tau, t_\mathrm{w}) \sim f_q(t_\mathrm{w}) \tau^{-\nu_\mathrm{s}/2} \quad \tau \gg t_\mathrm{w} \simeq 1, \tag{19}$$

where $f_q(t_\mathrm{w}) \simeq 1$ for $t_\mathrm{w} \sim 1$, and for the response

$$r(\tau, t_\mathrm{w}) \simeq f_r(t_\mathrm{w}) \tau^{-\nu_\mathrm{s}/2} \quad \tau \gg t_\mathrm{w} \simeq 1, \tag{20}$$

where again $f_r(t_\mathrm{w}) \sim 1$.

We are, however, mainly interested in the case of large waiting times $t_\mathrm{w} \gg 1$ and large time separations $\tau \gg 1$. In this case we can insert the asymptotic expansions (17, 18) in expressions (11, 12) and get for $\tau, t_\mathrm{w} \gg 1$

$$q(\tau, t_\mathrm{w}) \simeq \frac{\left(1 + \frac{\tau}{t_\mathrm{w}}\right)^{\nu_\mathrm{s}/2}}{\left(1 + \frac{\tau}{2t_\mathrm{w}}\right)^{\nu_\mathrm{s}}} \times \left[1 - \frac{1}{2} T \int_0^{2\tau} \mathrm{d}x \mathrm{e}^{-a_m x/2} \frac{f(x/4)}{\left(1 - \frac{x}{4t_\mathrm{w}(1+\tau/2t_\mathrm{w})}\right)^{\nu_\mathrm{s}}}\right] \tag{21}$$

for the leading asymptotic behaviour of the autocorrelation and

$$r(\tau, t_\mathrm{w}) \simeq b \left(1 + \frac{\tau}{t_\mathrm{w}}\right)^{\nu_\mathrm{s}/2} \tau^{-\nu_\mathrm{s}} \tag{22}$$

for the response. The prefactor b is $b^\mathrm{fm} = (4\pi)^{-\nu_\mathrm{s}}$ in the case of the ferromagnet and $b^\mathrm{sk} = (4\pi)^{1-\nu_\mathrm{s}}$ for the SK model. These equations show that the asymptotic dynamics in the limit $\tau, t_\mathrm{w} \gg 1$ of the spherical ferromagnet and the spherical SK model possesses the same *formal* structure. In particular we find that the scaling behaviour of the dynamical observables autocorrelation and response of the spherical ferromagnet in $d = 3$ and the spherical SK model is equivalent (on the level of exponents). This result was independently stated in [6]. The formal correspondence of the two models implies in particular that in both models the same characteristic time scales appear.

Before entering the discussion of the relevant time scales in the problem, let us simplify expression (21) further. Expanding the denominator of the integrand in this expression in a power series, one can prove that in the limit of large waiting times $t_\mathrm{w} \gg 1$ the dominant contribution to the integral comes for all times $\tau \gg 1$ from the zeroth order term of the expansion. Defining

$$q_\mathrm{p} := 1 - \frac{T}{2} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-a_m x/2} f(x/4) = 1 - \frac{T}{T_\mathrm{c}}, \tag{23}$$

where the last equality follows from (15), we find

$$q(\tau, t_\mathrm{w}) \simeq \frac{\left(1 + \frac{\tau}{t_\mathrm{w}}\right)^{\nu_\mathrm{s}/2}}{\left(1 + \frac{\tau}{2t_\mathrm{w}}\right)^{\nu_\mathrm{s}}} \left(q_\mathrm{p} + c_0 \tau^{1-\nu_\mathrm{s}}\right) \tag{24}$$

for the leading behaviour of the autocorrelation in the limit $\tau, t_\mathrm{w} \gg 1$, in which $c_0 = bT/(\nu_\mathrm{s} - 1)$.

Using (22) and (24) it is now straightforward to identify the different time scales of the problem. At first sight we find the two time scales already discussed in [9] for the case of the spherical SK model. The first is the time scale $t_0 \sim 1$ of the microscopic relaxation. At the upper end of this scale we have $\tau \gg 1$ but still $\tau/t_\mathrm{w} \ll 1$, such that we can neglect all waiting time dependent corrections. On this time scale the dynamics corresponds to the dynamics in equilibrium, *i.e.* it is time translation invariant with autocorrelation $q(\tau, t_\mathrm{w}) = \tilde{q}_0(\tau_0)$ and response $r(\tau, t_\mathrm{w}) = \tilde{r}_0(\tau_0)$ being functions of the scaling variable $\tau_0 := \tau/t_0$ only, and autocorrelation and response satisfy the FDT $-\partial_\tau \tilde{q}_0(\tau_0) = T\tilde{r}_0(\tau_0)$ of equilibrium dynamics. Therefore we will refer to this time scale as the FDT regime. At the upper end $\tau_0 \gg 1$ of this scale the response is found from (22) to be

$$r(\tau, t_\mathrm{w}) = \tilde{r}_0(\tau_0) \simeq \hat{b}_0 \tau_0^{-\nu_\mathrm{s}}, \tag{25}$$

with $\hat{b}_0 = b t_0^{-\nu_\mathrm{s}}$, while (24) implies for the correlation

$$q(\tau) = \tilde{q}_0(\tau_0) \simeq q_\mathrm{p} + \hat{c}_0 \tau_0^{1-\nu_\mathrm{s}} \tag{26}$$

with $\hat{c}_0 = c_0 t_0^{1-\nu_\mathrm{s}}$. This corresponds to a power law decay of the correlation to a plateau value q_p, which in the case of the spherical ferromagnet is just the square of the static spontaneous magnetization $\langle s_i \rangle^2$ [4], while in the case of the spherical SK model it is the static Edwards-Anderson parameter $q_\mathrm{EA} = \left[\langle s_i \rangle^2\right]_J$ [9,15]. The exponent of the decay is

$$\nu_0 := 1 - \nu_\mathrm{s} \tag{27}$$

which in the case of the spherical SK model is just the special case $p = 2$ of the result found in [8] for the equilibrium decay of the correlation of the spherical p-spin glass. If we speak of a plateau in the correlation, it is of course understood that this plateau in the autocorrelation is only visible in a plot against the logarithm of time τ.

The second obvious time scale is the waiting time itself. For $\tau \sim t_\mathrm{w}$ correlation and response can be written as functions of the scaling variable $\tau_\mathrm{w} := \tau/t_\mathrm{w}$ and one finds asymptotically for $\tau_\mathrm{w} \gg 1$ power law decays of the dynamical observables to zero [9].

From (22) it is obvious that these two are the only time scales which can be identified from the behaviour of the response function. However, it turns out that there exists a further nontrivial time scale in the problem, which can be identified from the autocorrelation function. Looking at expression (24) and taking into account the leading waiting time dependent correction in the prefactor we arrive at

$$q(\tau, t_w) \simeq \left(1 - \frac{\nu_s}{8}\left(\frac{\tau}{t_w}\right)^2\right)\left(q_p + c_0\tau^{1-\nu_s}\right). \tag{28}$$

This expression shows that there exists a waiting time dependent scale $t_p(t_w)$, on which the correlation begins to decay away from the plateau value q_p. To be more precise, we define this time scale t_p by requiring $q(t_p, t_w) = q_p$, such that t_p corresponds to the middle of the plateau of the correlation function. This means that t_p is the time for which the competing corrections in (28) are of the same order of magnitude. Hence we find that the time scale $t_p(t_w)$ scales as

$$t_p(t_w) \sim {t_w}^{2/(1+\nu_s)} \ll t_w \tag{29}$$

with the waiting time t_w. The latter inequality follows as $\nu_s > 1$. The plateau regime corresponding to time scale t_p is the so far missing link between the stationary dynamics within the FDT regime and the non-stationary dynamics for times of the order of the waiting time t_w itself. We will shortly see that it is in particular the time scale on which the FDT of equilibrium dynamics is violated.

In terms of the scaling variable $\tau_p := \tau/t_p$, the correlation within the plateau regime $\tau \sim t_p$ can be expressed in the scaling form

$$q(\tau, t_w) = q_p + \hat{q}_p(t_w)\tilde{q}_p(\tau_p) \tag{30}$$

with the prefactor $\hat{q}_p(t_w) = t_w^{-2(\nu_s-1)/(\nu_s+1)} \sim t_p^{\nu_0}$ and the scaling function

$$\tilde{q}_p(\tau_p) \simeq c_0\tau_p^{\nu_0} + c_p{\tau_p}^{\nu_1} \tag{31}$$

in which $c_p = -\frac{\nu_s}{8}q_p$ and $\nu_1 = 2$ (recall that $\nu_0 = 1 - \nu_s < 0$). This scaling function describes the decay of the correlation towards q_p at the lower end of the plateau scale, *i.e.* for $\tau_p \ll 1$, and its subsequent decay away from q_p at the upper end of the plateau scale where $\tau_p \gg 1$.

In order to study the violation of the FDT we introduce a quantity $n(\tau, t_w)$ which characterizes this violation quantitatively [12] *via*

$$-\partial_\tau q(\tau, t_w) =: T(1 + n(\tau, t_w))r(\tau, t_w). \tag{32}$$

Note that differentiating with respect to the time separation is equivalent to differentiating with respect to the later time t. Alternatively one may differentiate with respect to the earlier time t_w. In the representation of the correlation in terms of t_w and time difference $\tau = t - t_w$ this gives rise to the corresponding definition

$$\hat{\partial}_{t_w} q(\tau, t_w) =: T(1 + n_w(\tau, t_w))r(\tau, t_w), \tag{33}$$

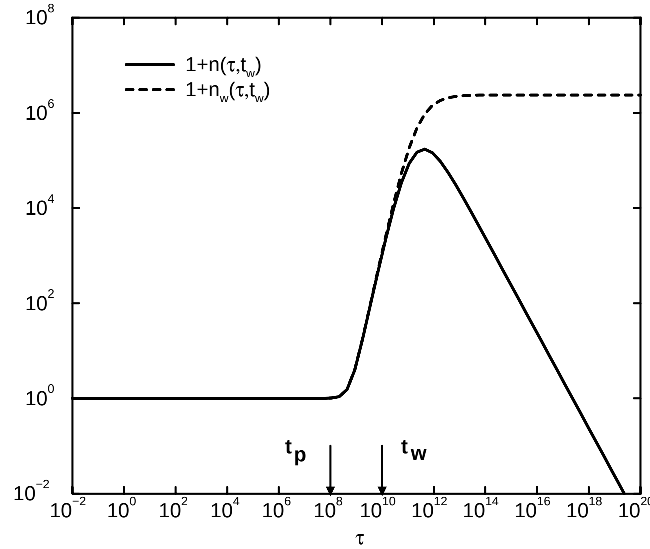

Fig. 1. $1 + n(\tau, t_w)$ and $1 + n_w(\tau, t_w)$ as functions of τ for $t_w = 10^{10}$. Vertical arrows mark the plateau scale $t_p(t_w)$ and the waiting time scale t_w.

with $\hat{\partial}_{t_w} = \partial_{t_w} - \partial_\tau$. The latter is related to the fluctuation dissipation ratios $X(\tau, t_w)$ studied *e.g.* in [6,11] *via* $X(\tau, t_w) = 1/(1 + n_w(\tau, t_w))$. For $\tau, t_w \gg 1$ we get

$$\begin{aligned} 1 + n(\tau, t_w) &\simeq \frac{1}{\left(1 + \frac{\tau}{2t_w}\right)^{\nu_s}} \\ &\quad \times \left[1 + \frac{\nu_s q_p}{4bT}\frac{\tau^{1+\nu_s}}{t_w^2}\frac{1}{\left(1 + \frac{\tau}{t_w}\right)\left(1 + \frac{\tau}{2t_w}\right)}\right], \end{aligned} \tag{34}$$

$$\begin{aligned} 1 + n_w(\tau, t_w) &\simeq \frac{1}{\left(1 + \frac{\tau}{2t_w}\right)^{\nu_s}} \\ &\quad \times \left[1 + \frac{\nu_s q_p}{4bT}\frac{\tau^{1+\nu_s}}{t_w^2}\frac{1}{\left(1 + \frac{\tau}{2t_w}\right)}\right]. \end{aligned} \tag{35}$$

Figure 1 shows these two functions for $T = 0.6\,T_c$, and $t_w = 10^{10}$ so that $t_p = 10^8$. As we have seen before, the decay towards the plateau satisfies the FDT, that is, we have in leading order $n(\tau, t_w) = n_w(\tau_t w) = 0$ for $\tau \ll t_p$. On the intermediate time scale $t_p = t_w^{2/(1+\nu_s)}$, however, we obtain the scaling form

$$n(\tau, t_w) \simeq n_w(\tau, t_w) \simeq \tilde{n}(\tau_p) = \frac{\nu_s q_p}{4bT}\,{\tau_p}^{1+\nu_s}, \tag{36}$$

which approaches zero at the lower end of the plateau scale (where the FDT holds) but is non-zero (indicating FDT violation) for all $\tau_p = \mathcal{O}(1)$, and exhibits a power law divergence at the upper end of the t_p scale. This can be traced back to the fact that the behaviour of the response function $r(\tau, t_w)$ does not change on the time scale t_p whereas that of the correlation does. It is this divergence which is responsible for the fact noted in [6,11] that parametric representations of integrated response $\chi(\tau, t_w) = \int_0^\tau \mathrm{d}s\, r(s, t_w + \tau - s)$ *versus* correlation $q(\tau, t_w)$ saturate at the value $\chi = (1 - q_p)/T$ for

$q(\tau, t_\mathrm{w}) \leq q_\mathrm{p}$ for the models considered in the present paper. Note that the nature of this divergence is in the large t_w limit of course not detectable in such parametric plots, as it occurs entirely on the plateau scale, on which $q(\tau, t_\mathrm{w})$ is basically arrested at q_p. It may however be obtained from the finite t_w-corrections to such plots which may be extracted from (34) and (35).

Note that (34) and (35) imply that n and n_w exhibit *different* scaling on the t_w scale. With $\tau_\mathrm{w} = \tau/t_\mathrm{w}$ we have

$$1 + n(\tau, t_\mathrm{w}) \simeq \frac{1}{(1+\tau_\mathrm{w}/2)^{\nu_\mathrm{s}}} \times \left[1 + \frac{\nu_\mathrm{s} q_\mathrm{p}}{4bT} t_\mathrm{w}^{\nu_\mathrm{s}-1} \tau_\mathrm{w}^{1+\nu_\mathrm{s}} \frac{1}{(1+\tau_\mathrm{w})(1+\tau_\mathrm{w}/2)}\right], \quad (37)$$

$$1 + n_\mathrm{w}(\tau, t_\mathrm{w}) \simeq \frac{1}{(1+\tau_\mathrm{w}/2)^{\nu_\mathrm{s}}} \times \left[1 + \frac{\nu_\mathrm{s} q_\mathrm{p}}{4bT} t_\mathrm{w}^{\nu_\mathrm{s}-1} \tau_\mathrm{w}^{1+\nu_\mathrm{s}} \frac{1}{(1+\tau_\mathrm{w}/2)}\right], \quad (38)$$

implying that both are infinite in the $t_\mathrm{w} \to \infty$ limit, but show different behaviour at large τ_w when t_w is large but finite: Whereas $1 + n(\tau, t_\mathrm{w}) \simeq \frac{\nu_\mathrm{s} q_\mathrm{p}}{bT} (2t_\mathrm{w})^{\nu_\mathrm{s}-1} \tau_\mathrm{w}^{-1} \to 0$ as $\tau_\mathrm{w} \to \infty$, we have $1 + n_\mathrm{w}(\tau, t_\mathrm{w}) \simeq \frac{\nu_\mathrm{s} q_\mathrm{p}}{bT} (2t_\mathrm{w})^{\nu_\mathrm{s}-1}$ in the same limit.

To summarize, the FDT is broken already on the time scale t_p rather than only on the scale t_w, the former being much smaller than the latter when t_w becomes large, since $t_\mathrm{p}/t_\mathrm{w} \to 0$ as $t_\mathrm{w} \to \infty$. Moreover, the divergence of n and n_w on the t_p scale implies that the QFDT solution, which was found in [8] for the spherical p-spin glass with $p > 2$ and in [14] for manifolds in disordered potentials, does not exist in the case of the spherical SK model (and in the ferromagnetic systems). From (28) it is also obvious that for the models considered here the plateau regime t_p is the only further time scale in the problem. This, too, is in contrast to the expectations for spherical p-spin glass with $p > 2$, which will be considered in a forthcoming paper [13].

5 Discussion

Considering the simplicity of the models discussed in the previous sections, the complexity of the dynamical behaviour seems rather astonishing. However, in the case of spherical ferromagnet the explicit dependence on the waiting time of both correlation and response even in the limit $\tau \gg t_\mathrm{w} \gg 1$ is a well-known result in the theory of phase ordering kinetics [7]. According to the scaling hypothesis of coarsening dynamics there exists for large times $t = \tau + t_\mathrm{w} \gg 1$ a single length scale $L(t)$ in the system, which can be interpreted as the typical size of a domain at time t. This means that for large times $t \gg t_\mathrm{w} \gg 1$ the two-time-autocorrelation function of such a system is a function of the ratio of the two length scales $L(t) \gg L(t_\mathrm{w}) \gg 1$ only. The exact solution of the phase ordering dynamics of the spherical ferromagnet yields for the autocorrelation at $T = 0$ in the limit of large times the result [7]

$$q(t - t_\mathrm{w}, t_\mathrm{w}) = \left(\frac{4tt_\mathrm{w}}{(t+t_\mathrm{w})^2}\right)^{d/4}, \quad (39)$$

which is just equation (21) for $T = 0$.

It has been noted that aging behaviour in the correlation functions of coarsening systems has a simple interpretation in terms of domain growth [5–7,11]. This holds in particular for the emergence of the plateau scale. After the system has spent the waiting time $t_\mathrm{w} \gg 1$ in the low temperature phase, an arbitrary spin will on average be found in a domain of size $L(t_\mathrm{w})$. The autocorrelation of this spin will for short times decay towards q_p, which is the square of the local spontaneous magnetization, due to spin fluctuations within this domain. This decay is equivalent to a decay within a local equilibrium state and satisfies time translational invariance and FDT. The further asymptotic decay of the autocorrelation away from this value q_p towards zero can only be produced by a change of the environment of the chosen spin, which means that a domain wall has to pass by its site. As the size of the original domain grows as a power of the waiting time, it is very plausible that the time spent near the plateau value should also grow as a power of the waiting time. Since the growth of the domains and therefore the wandering of the domain walls is a slow process the asymptotic decay towards zero is also a slow power law decay. For the spherical SK model, such arguments are of course not available, as the model does not possess a geometry.

At the heart of it, the *formal* equivalence of the asymptotic dynamics for $\tau, t_\mathrm{w} \gg 1$ of the spherical ferromagnet and the spherical SK model is due to the fact that both interaction matrices exhibit eigenvalue densities $\rho(a)$ whose behaviour at the upper end a_m of the spectrum can be characterized by a power law

$$\rho(a) \sim (a_m - a)^{\nu_\mathrm{s}-1}, \quad \text{as} \quad a \to a_m, \quad (40)$$

with the exponent ν_s introduced earlier. It is this feature which determines the behaviour of correlation and response in these systems at $\tau, t_\mathrm{w} \gg 1$. The origin of the power law may be disorder, as in the case of the SK model and the semi-circle law, but it need not, as exemplified by the d-dimensional ordered systems. Thus aging in the spherical SK model cannot be interpreted as spin glass aging as it is observed experimentally [1,2,16] as well as in model calculations [10] and simulations [18]. Indeed, it is well known that from a static point of view this model does not have the properties of a typical spin glass, as it has a replica symmetric solution for all temperatures and does not possess many degenerate ground states. The results above imply that the spherical SK model is neither a spin glass from a dynamical point of view, despite the existence of a plateau in the correlation function as it is observed in spin glasses and related systems [8,12].

Obviously the autocorrelation is not a suitable quantity to distinguish aging in a spin glass from the simpler case of coarsening dynamics in magnetic systems, whose nonequlibrium dynamics is determined by domain growth. A dynamical observable which characterizes a spin glass, however, is given by the thermoremanent magnetization defined in (7). This is due to the fact that it is the response function which is most sensitive to the complex phase space structure exhibited by spin glasses. To be more precise, the particular metastable configurations of a spinglass depend strongly on a magnetic field. During the waiting time the system is expected to move to configurations of increasing stability. On the other hand, a state which is relatively stable in a given field might become less stable if the field is slightly changed. This means that, after a change of the field at t_{w}, the system has to move to new states of increasing stability. The time scale of this process depends on the degree of stability reached at t_{w}. This leads to a plateau in the thermoremanent magnetization similar to the one found in the correlation function. This will be derived for the spherical p-spin glass with $p > 2$ in a forthcoming paper [13]. A mechanism of this kind is of course absent in a coarsening system and as a consequence $m_r(\tau, t_{\mathrm{w}})$ decays in the limit of large waiting times $t_{\mathrm{w}} \gg 1$ for all $\tau \ll t_{\mathrm{w}}$ as

$$m_r(\tau, t_{\mathrm{w}}) \sim \tau^{1-\nu_{\mathrm{s}}}. \tag{41}$$

To prove this result, let us denote by $t_1(t_{\mathrm{w}})$ a lower bound of the waiting time scale satisfying $\tau \ll t_1 \ll t_{\mathrm{w}}$. Let us further choose a time t_2, such that $t_{\mathrm{w}} - t_2 \sim 1$. Then we can split the integration in (7) as follows

$$m_r(\tau, t_{\mathrm{w}}) \simeq h \left(\int_0^{t_1} \mathrm{d}s\, r(\tau + s, t_{\mathrm{w}}) + \int_{t_1}^{t_2} \mathrm{d}s\, r(s, t_{\mathrm{w}} - s) + \int_{t_2}^{t_{\mathrm{w}}} \mathrm{d}s\, r(s, t_{\mathrm{w}} - s) \right). \tag{42}$$

Using (25) in the first integral we find that this term yields the leading order contribution given in (41) as the contribution from the upper bound is negligible in the limit $t_{\mathrm{w}} \gg 1$. In the last integral in (42) the argument $t_{\mathrm{w}} - s$ is always of order unity and with (20) we find that it scales as ${t_{\mathrm{w}}}^{-\nu_{\mathrm{s}}/2}$ with the waiting time, such that it is negligible in the limit of large waiting times. Thus we just have to consider the contributions from the middle of the integration range for the remaining integral in (42). Rewriting this integral in terms of the scaling variable $\sigma := s/t_{\mathrm{w}}$ we get using (22) that this term scales as ${t_{\mathrm{w}}}^{1-\nu_{\mathrm{s}}}$ which leads to (41) as the dominant contribution. Hence we have indeed found that in the type of models considered here the thermoremanent magnetization does not exhibit a plateau in the limit $t_{\mathrm{w}} \gg 1$ nor does it depend on the waiting time for all times $t \ll t_{\mathrm{w}}$. For coarsening systems this is what we expected as this relaxation stems from spin fluctuations within a certain domain, which do not know anything about the waiting time. As noted in [6,11] an alternative criterion to distinguish aging in coarsening systems from spin glass aging is the integrated response $\chi(\tau, t_{\mathrm{w}})$ mentioned in Section 4. Both, the saturation of $\chi(\tau, t_{\mathrm{w}})$ and the absence of a plateau in the thermoremanent magnetization are due to the same reason, namely due to the divergence of the function $n(\tau, t_{\mathrm{w}})$, equivalently due to the vanishing of the fluctuation dissipation ratio $X(\tau, t_{\mathrm{w}})$ on the plateau scale.

Let us finally stress that the time scale t_{p} also appears in the more complicated case of the spherical p-spin glass with $p > 2$, the spherical SK model being just the simplest of this class of models, and it is the behaviour of correlation and response on this time-scale which is needed to uniquely fix the dynamics at later times. This will be explicitly shown in a forthcoming paper [13].

It is a pleasure to thank H. Kinzelbach for numerous illuminating discussions.

References

1. M. Alba, M. Ocio, J. Hamman, J. Phys. Lett. France **46**, L-1101 (1985); Europhys. Lett. **2**, 45 (1986).
2. M. Alba, J. Hamman, M. Ocio, Ph. Refregier, H. Bouchiat, J. Appl. Phys. **61**, 3683 (1987); Ph. Refregier, M. Ocio, J. Hamman, E. Vincent, J. Appl. Phys. **63**, 4343 (1988).
3. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).
4. T.H. Berlin, M. Kac, Phys. Rev. **86**, 821 (1952).
5. L. Berthier, J.L. Barrat, J. Kurchan (preprint `cond-mat/9903091`).
6. J.P. Bouchaud, L.F. Cugliandolo, J. Kurchan, M. Mézard, *Out of Equilibrium Dynamics in Spin-Glasses and Other Glassy Systems*, in *Spin Glasses and Random Fields*, edited by A.P. Young (World Scientific, 1998).
7. A.J. Bray, Adv. Phys. **43**, 357 (1994).
8. A. Crisanti, H. Horner, H.-J. Sommers, Z. Phys. B **92**, 257 (1993).
9. L.F. Cugliandolo, D.S. Dean, J. Phys. A **28**, 4213 (1995).
10. L.F. Cugliandolo, J. Kurchan, Phys. Rev. Lett. **71**, 173 (1993).
11. L.F. Cugliandolo, J. Kurchan, L. Peliti, Phys. Rev. E **55**, 3898 (1997).
12. H. Horner, Z. Phys. B **100**, 243 (1996).
13. H. Horner, R. Kühn, W. Zippold (in preparation).
14. H. Kinzelbach, H. Horner, J. Phys. I France **3**, 1329 (1993); J. Phys. I France **3**, 1901 (1993).
15. J.M. Kosterlitz, D.J. Thouless, R.C. Jones, Phys. Rev. Lett. **36**, 1217 (1976).
16. L. Lundgren, P. Svedlindh, P. Nordblad, O. Beckman, Phys. Rev. Lett. **51**, 911 (1983); P. Nordblad, P. Svedlindh, L. Lundgren, L. Sandlund, Phys. Rev. B **33**, 645 (1986).
17. M.L. Mehta, *Ramdom Matrices* (Academic Press, 1991).
18. H. Rieger, in *Annual Reviews of Computational Physics*, edited by D. Stauffer (World Scientific, Singapore, 1995), p. 295.